Meaningful Generalization of the Exponent

Formal Outline: Course in Introductory Calculus Required

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1. Abstract

The exponent: enter a^n into a calculator, and the output will be $a \cdot a \cdot \ldots\ n$ times. But, what if the expression 2^π were entered into a calculator? Why does an actual output result? How is a meaningful value to be drawn out from an expression like 2^π?

2. Definition

$$\ln x = \int_1^x \frac{1}{s}\mathrm{d}s \qquad 0 < x$$

...Strictly increasing invertible function (derivative always positive for 0 < x):

$$\exp = \text{inverse} \ln$$

$$\Rightarrow x = \ln(\exp x)$$

$$x = \exp(\ln x) \qquad 0 < x$$

partial differentiation: differentiation with respect to one variable, with all other variables treated as constant

factorial: $n! = 1 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \ldots \cdot n$

integer conversion: $\text{int}(x) = \begin{pmatrix}\text{real number } x \text{ converted to integer} \\ \text{by discarding all digits after decimal}\end{pmatrix}$

absolute value: $|x| = \begin{cases} x & 0 \le x \\ -x & x < 0 \end{cases}$

3. Simple Exponent Definition

The purpose of the exponent is to reiterate multiplication a given number of times.

A meaningful output results from any real base $a \neq 0$ raised to the power of any integer input:

- Multiplication a positive number of times:

$$a^n = 1 \cdot \overbrace{a \cdot a \cdot \ldots}^{n \text{ times}}$$

- Multiplication zero number of times:

$$a^0 = 1 \qquad a \neq 0$$

- Multiplication a negative number of times translates to the inverse operation of multiplication:

$$a^{-n} = 1 \div \overbrace{a \div a \div \ldots}^{n \text{ times}} \qquad a \neq 0$$

4. Generalized Exponent Definition

The exponent can be generalized to output a real value from any real input as long as base a is restricted to positive real numbers:

$$\text{output} = a^{\text{input}} \qquad 0 < a$$

A defining property is drawn out directly from the simple exponent definition:

$$a^{n+m} = \overbrace{a \cdot a \cdot \dots}^{n \text{ times}} \cdot \overbrace{a \cdot a \cdot \dots}^{m \text{ times}} = a^n a^m$$

The three defining properties allow the exponent's output to be populated from any integer input <u>without</u> restricting the input to integers:

a. $a^1 = a$ $\qquad 0 < a$

b. $a^{-1} = 1 \div a$ $\qquad 0 < a$

c. $a^{x+y} = a^x a^y$ for all real numbers x, y $\qquad 0 < a$

The inclusion of three generalization properties produces one unique function of real input for the generalized exponent with positive real base a:

d. The generalized exponent outputs a positive real value and is differentiable with respect to input.

e. If $a \neq 1$, given any positive real output, a unique inverse exists as input and has a positive derivative.

f. Given any real input, $1 = 1^{\text{input}}$.

5. Meaningful Generalization of the Exponent

a. **Case $a \neq 1$:**

...Function defined as generalized exponent:

$f_a(\text{input}) = a^{\text{input}}$ $\qquad 0 < a$

$\text{output} = f_a(\text{input})$ $\qquad 0 < a$

*...Application of property (**3. d.**):*

$0 < \text{output}$

*...Application of property (**3. e.**):*

$f_a^{-1}(\text{output}) = \text{input}$ $\qquad 0 < a$

... $\text{output} = f_a(\text{input})$ $\qquad 0 < a$

$\qquad = f_a\left(f_a^{-1}(\text{output})\right)$ $\qquad 0 < a$

Positive real numbers x, y selected as output variables:

$0 < x$

$0 < y$

$x = f_a\left(f_a^{-1}(x)\right)$ $\qquad 0 < a$

$y = f_a\left(f_a^{-1}(y)\right)$ $\qquad 0 < a$

*...Application of property (**3. c.**):*

$$f_a\left(f_a^{-1}(x) + f_a^{-1}(y)\right) = f_a\left(f_a^{-1}(x)\right) f_a\left(f_a^{-1}(y)\right) \qquad 0 < a$$

$$= xy \qquad 0 < a$$

*...Partial differentiation (**3. d.** and **3. e.**)*

with respect to output variable x:

$$f_a'\left(f_a^{-1}(x) + f_a^{-1}(y)\right) (f_a^{-1})'(x) = y \qquad 0 < a$$

$$f_a'\left(f_a^{-1}(x) + f_a^{-1}(y)\right) = \frac{y}{(f_a^{-1})'(x)} \qquad 0 < a$$

*...Partial differentiation (**3. d.** and **3. e.**)*

with respect to output variable y:

$$f_a'\left(f_a^{-1}(x) + f_a^{-1}(y)\right) (f_a^{-1})'(y) = x \qquad 0 < a$$

$$f_a'\left(f_a^{-1}(x) + f_a^{-1}(y)\right) = \frac{x}{(f_a^{-1})'(y)} \qquad 0 < a$$

...Both results equated:

$$\frac{y}{(f_a^{-1})'(x)} = \frac{x}{(f_a^{-1})'(y)} \qquad 0 < a$$

$$(f_a^{-1})'(y)\, y = (f_a^{-1})'(x)\, x \qquad 0 < a$$

...No dependence implied on either output variable:

$$(f_a^{-1})'(y)\, y = \text{constant}_a \qquad 0 < a$$

$$(f_a^{-1})'(x)\, x = \text{constant}_a \qquad 0 < a$$

$$(f_a^{-1})'(\text{output})\, \text{output} = \text{constant}_a \qquad 0 < a$$

$$(f_a^{-1})'(\text{output}) = \frac{\text{constant}_a}{\text{output}} \qquad 0 < a$$

...Antidifferentiation with respect to output

utilizing dummy variable of integration s:

$$f_a^{-1}(\text{output}) = \int_1^{\text{output}} \frac{\text{constant}_a}{s} \mathrm{d}s + \mathrm{c} \qquad 0 < a$$

$$f_a^{-1}(1) = \int_1^{1} \frac{\text{constant}_a}{s} \mathrm{d}s + \mathrm{c} \qquad 0 < a$$

$$= \mathrm{c} \qquad 0 < a$$

...Value of input that results in output = 1:

$$a^0 = a^{1-1} \overset{\textit{3.c.}}{=} a^1 a^{-1} \overset{\textit{3.b.}}{=} a^1 (1 \div a) \overset{\textit{3.a.}}{=} a(1 \div a) = 1 \qquad 0 < a$$

$$f_a(0) = 1 \qquad 0 < a$$

$$0 = f_a^{-1}(1) \qquad 0 < a$$

$$= \mathrm{c}$$

…Substitution for constant c and ln definition back into previous equation:

… $f_a^{-1}(\text{output}) = \text{constant}_a \ln(\text{output})$ $0 < a$

$\text{input} = \text{constant}_a \ln(\text{output})$ $0 < a$

…Generalized exponent well-defined (inverse not everywhere equal to 0):

$\text{constant}_a \neq 0$ $0 < a$

…Equation solved for output:

$\text{output} = \exp\left(\frac{\text{input}}{\text{constant}_a}\right)$ $0 < a$

$f_a(\text{input}) = \exp\left(\frac{\text{input}}{\text{constant}_a}\right)$ $0 < a$

$f_a(1) = \exp\left(\frac{1}{\text{constant}_a}\right)$ $0 < a$

*…Application of property (**3. a.**):*

$a = \exp\left(\frac{1}{\text{constant}_a}\right)$ $0 < a$

$\ln a = \frac{1}{\text{constant}_a}$ $0 < a$

…Substitution back into previous equation:

… $f_a(\text{input}) = \exp(\text{input} \ln a)$ $0 < a$

- $a^{\text{input}} = \exp(\text{input} \ln a)$ $0 < a$

b. Case $\boldsymbol{a = 1}$:

*…Application of property (**3. f.**):*

$1^{\text{input}} = 1$

…Composition of inverse functions:

$= \exp(\ln 1)$

…Substitution for ln definition:

$= \exp\left(\int_1^1 \frac{1}{s} \mathrm{d}s\right)$

$= \exp(0)$

$= \exp(\text{input} \cdot 0)$

$= \exp(\text{input} \ln 1)$

- $a^{\text{input}} = \exp(\text{input} \ln a)$ $0 < a$

6. Expression for ln

a. Infinite Series

...Only simple exponents found in each term:

$$y = 1 + x^1 + x^2 + x^3 + \cdots + x^n$$

$$xy = x^1 + x^2 + x^3 + x^4 + \cdots + x^{n+1}$$

$$y - xy = 1 - x^{n+1}$$

$$y = \frac{1 - x^{n+1}}{1 - x}$$

$$\frac{1 - x^{n+1}}{1 - x} = 1 + x^1 + x^2 + x^3 + \cdots + x^n$$

...Terms vanish as n approaches infinity:

- $\frac{1}{1 - x} = 1 + x^1 + x^2 + x^3 + \cdots$ $\quad -1 < x < 1$
- $\frac{1}{1 - x^2} = 1 + x^2 + x^4 + x^6 + \cdots$ $\quad -1 < x < 1$

b. Proposed Infinite Series

...Only simple exponents found in each term:

- $-\frac{2}{1}x^1 - \frac{2}{3}x^3 - \frac{2}{5}x^5 - \frac{2}{7}x^7 - \cdots$ $\quad -1 < x < 1$

c. Condition: Proposed Infinite Series Results in a Single Finite Value

...Factored out the even infinite series:

$$-\frac{2}{1}x^1 - \frac{2}{3}x^3 - \frac{2}{5}x^5 - \frac{2}{7}x^7 - \cdots = -2x\left(1 + \frac{x^2}{3} + \frac{x^4}{5} + \frac{x^6}{7} + \cdots\right) \qquad -1 < x < 1$$

...Term-by-term comparisons:

$$\frac{x^2}{3} \le x^2$$

$$\frac{x^4}{5} \le x^4$$

$$\frac{x^6}{7} \le x^6$$

$$\ldots \le \ldots$$

...Implied inequality:

$$1+\frac{x^2}{3}+\frac{x^4}{5}+\frac{x^6}{7}+\cdots \le 1+x^2+x^4+x^6+\cdots \qquad -1<x<1$$

*...Substitution for infinite series (**6. a.**):*

$$1+\frac{x^2}{3}+\frac{x^4}{5}+\frac{x^6}{7}+\cdots \le \frac{1}{1-x^2} \qquad -1<x<1$$

...Nondecreasing with each successive term

and existence of a finite upper bound:

$$1+\frac{x^2}{3}+\frac{x^4}{5}+\frac{x^6}{7}+\cdots = \text{(finite value)} \qquad -1<x<1$$

...Substitution back into previous equation:

$$\ldots \quad -\frac{2}{1}x^1-\frac{2}{3}x^3-\frac{2}{5}x^5-\frac{2}{7}x^7-\cdots = -2x\,\text{(finite value)} \qquad -1<x<1$$

Proposed infinite series results in a single finite value:

- $\rho(x) = -\frac{2}{1}x^1-\frac{2}{3}x^3-\frac{2}{5}x^5-\frac{2}{7}x^7-\cdots \qquad -1<x<1$

d. Expression for ln

...Differentiation with respect to x:

$$\frac{\mathrm{d}}{\mathrm{d}x}\rho(x) = -2-2x^2-2x^4-2x^6-\cdots \qquad -1<x<1$$

*...Substitution for infinite series (**6. a.**):*

$$= \frac{-2}{1-x^2} \qquad -1<x<1$$

...Insertion of an input transformation

and differentiation with respect to x:

$$\frac{\mathrm{d}}{\mathrm{d}x}\rho\left(\frac{1-x}{1+x}\right) = \left(\frac{-2}{1-(1-x)^2/(1+x)^2}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1-x}{1+x}\right) \qquad -1<\frac{1-x}{1+x}<1$$

$$= \left(\frac{-2}{1-(1-x)^2/(1+x)^2}\right)\left(\frac{-(1+x)-(1-x)}{(1+x)^2}\right) \qquad -1<\frac{1-x}{1+x}<1$$

$$= \frac{4}{(1+x)^2-(1-x)^2} \qquad -1<\frac{1-x}{1+x}<1$$

$$= \frac{4}{(1+2x+x^2)-(1-2x+x^2)} \qquad -1<\frac{1-x}{1+x}<1$$

$$= \frac{4}{4x} \qquad -1<\frac{1-x}{1+x}<1$$

$$= \frac{1}{x} \qquad -1<\frac{1-x}{1+x}<1$$

...Equivalent inequalities:

$$-1 < \frac{1-x}{1+x} < 1$$

$$\Leftrightarrow \begin{pmatrix} 0 < 1+x \\ -1-x < 1-x < 1+x \end{pmatrix} \text{ or } \begin{pmatrix} 1+x < 0 \\ -1-x > 1-x > 1+x \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} -1 < x \\ -2 < 0 < 2x \end{pmatrix} \text{ or } \begin{pmatrix} x < -1 \\ -2 > 0 > 2x \end{pmatrix}$$

$$\Leftrightarrow 0 < x$$

...Substitution for equivalent inequality back into previous equation:

$$\ldots \quad \frac{\mathrm{d}}{\mathrm{d}x}\rho\left(\frac{1-x}{1+x}\right) = \frac{1}{x} \qquad 0 < x$$

...Antidifferentiation with respect to x utilizing dummy variable of integration s:

$$\rho\left(\frac{1-x}{1+x}\right) = \int_1^x \frac{1}{s}\,\mathrm{d}s + \mathrm{c} \qquad 0 < x$$

...Substitution for ln definition:

$$\rho\left(\frac{1-x}{1+x}\right) = \ln x + \mathrm{c} \qquad 0 < x$$

$$\ln x = \rho\left(\frac{1-x}{1+x}\right) - \mathrm{c} \qquad 0 < x$$

$$= -\mathrm{c} - \frac{2}{1}\left(\frac{1-x}{1+x}\right)^1 - \frac{2}{3}\left(\frac{1-x}{1+x}\right)^3 - \frac{2}{5}\left(\frac{1-x}{1+x}\right)^5 - \frac{2}{7}\left(\frac{1-x}{1+x}\right)^7 - \cdots \qquad 0 < x$$

$$\ln 1 = -\mathrm{c}$$

...Substitution for ln definition:

$$\int_1^1 \frac{1}{s}\,\mathrm{d}s = -\mathrm{c}$$

$$0 = \mathrm{c}$$

...Substitution for constant c back into previous equation:

- $\ln x = -\frac{2}{1}\left(\frac{1-x}{1+x}\right)^1 - \frac{2}{3}\left(\frac{1-x}{1+x}\right)^3 - \frac{2}{5}\left(\frac{1-x}{1+x}\right)^5 - \frac{2}{7}\left(\frac{1-x}{1+x}\right)^7 - \cdots \qquad 0 < x$

7. Expression for exp

a. Key Infinite Series

...Only simple exponents found in each term:

- $1+\frac{x^1}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$

b. Condition: Key Infinite Series Results in a Single Finite Value

Key infinite series split at n^th^ term, where $n=|\mathrm{int}(x)|$:

...Initial portion as finite series (when 0 < n)

and remainder portion as infinite series:

$$1+\frac{x^1}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots=1+\frac{x^1}{1}+\frac{x^2}{1\cdot 2}+\frac{x^3}{1\cdot 2\cdot 3}+\cdots+\frac{x^{n-1}}{(n-1)!}$$
$$+\frac{x^n}{n!}+\frac{x^{n+1}}{(n+1)!}+\frac{x^{n+2}}{(n+2)!}+\frac{x^{n+3}}{(n+3)!}+\frac{x^{n+4}}{(n+4)!}+\frac{x^{n+5}}{(n+5)!}+\cdots$$

...Remainder portion separated

into even/odd infinite series:

$$=1+\frac{x^1}{1}+\frac{x^2}{1\cdot 2}+\frac{x^3}{1\cdot 2\cdot 3}+\cdots+\frac{x^{n-1}}{(n-1)!}$$
$$+\frac{x^n}{n!}+\frac{x^{n+2}}{(n+2)!}+\frac{x^{n+4}}{(n+4)!}+\cdots$$
$$+\frac{x^{n+1}}{(n+1)!}+\frac{x^{n+3}}{(n+3)!}+\frac{x^{n+5}}{(n+5)!}+\cdots$$

...Even/odd infinite series factored:

$$=1+\frac{x^1}{1}+\frac{x^2}{1\cdot 2}+\frac{x^3}{1\cdot 2\cdot 3}+\cdots+\frac{x^{n-1}}{(n-1)!}$$
$$+\frac{x^n}{n!}\left(1+\frac{x^2}{(n+1)(n+2)}+\frac{x^4}{(n+1)(n+2)(n+3)(n+4)}+\cdots\right)$$
$$+\frac{x^{n+1}}{(n+1)!}\left(1+\frac{x^2}{(n+2)(n+3)}+\frac{x^4}{(n+2)(n+3)(n+4)(n+5)}+\cdots\right)$$

...Key inequality:

$$-1<\frac{x}{n+1}<1$$

…Term-by-term comparisons:

$$\frac{x^2}{(n+1)(n+2)} \le \left(\frac{x}{n+1}\right)^2$$

$$\frac{x^4}{(n+1)(n+2)(n+3)(n+4)} \le \left(\frac{x}{n+1}\right)^4$$

$$\dots \le \dots$$

$$\frac{x^2}{(n+2)(n+3)} \le \left(\frac{x}{n+1}\right)^2$$

$$\frac{x^4}{(n+2)(n+3)(n+4)(n+5)} \le \left(\frac{x}{n+1}\right)^4$$

$$\dots \le \dots$$

…Implied inequalities:

$$1 + \frac{x^2}{(n+1)(n+2)} + \frac{x^4}{(n+1)(n+2)(n+3)(n+4)} + \cdots \le 1 + \left(\frac{x}{n+1}\right)^2 + \left(\frac{x}{n+1}\right)^4 + \cdots$$

$$1 + \frac{x^2}{(n+2)(n+3)} + \frac{x^4}{(n+2)(n+3)(n+4)(n+5)} + \cdots \le 1 + \left(\frac{x}{n+1}\right)^2 + \left(\frac{x}{n+1}\right)^4 + \cdots$$

*…Substitution for infinite series (**6. a.**):*

$$1 + \frac{x^2}{(n+1)(n+2)} + \frac{x^4}{(n+1)(n+2)(n+3)(n+4)} + \cdots \le \frac{1}{1-(x/(n+1))^2}$$

$$1 + \frac{x^2}{(n+2)(n+3)} + \frac{x^4}{(n+2)(n+3)(n+4)(n+5)} + \cdots \le \frac{1}{1-(x/(n+1))^2}$$

…Nondecreasing with each successive term

and existence of a finite upper bound:

$$1 + \frac{x^2}{(n+1)(n+2)} + \frac{x^4}{(n+1)(n+2)(n+3)(n+4)} + \cdots = (\text{finite value \#1})$$

$$1 + \frac{x^2}{(n+2)(n+3)} + \frac{x^4}{(n+2)(n+3)(n+4)(n+5)} + \cdots = (\text{finite value \#2})$$

…Substitution back into previous equation:

$$\dots \quad 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = 1 + \frac{x^1}{1} + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \cdots + \frac{x^{n-1}}{(n-1)!}$$

$$+ \frac{x^n}{n!}(\text{finite value \#1}) + \frac{x^{n+1}}{(n+1)!}(\text{finite value \#2})$$

Key infinite series results in a single finite value:

- $k(x) = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

c. **Key Properties**

$$k(x) = 1 + \frac{x^1}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \cdots$$

$$k'(x) = 1 + \frac{x^1}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots$$

i. $k'(x) = k(x)$

... $$k(x) = 1 + \frac{x^1}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \cdots$$

$$k(-x) = 1 - \frac{x^1}{1} + \frac{x^2}{1 \cdot 2} - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \cdots$$

...Strictly increasing as x increases:

$$\frac{k(x) + k(-x)}{2} = 1 + \frac{x^2}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots$$

...Strictly increasing as x increases:

$$\frac{k(x) - k(-x)}{2} = \frac{x^1}{1} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \cdots$$

...Sum of two strictly increasing functions
results in a strictly increasing function:

$$k(x) = \frac{k(x) + k(-x)}{2} + \frac{k(x) - k(-x)}{2}$$

...Strictly increasing as x increases
(derivative always positive):

$$0 < k'(x)$$

*...Application of key property (**i**):*

ii. $0 < k(x)$

... $$k(x) = 1 + \frac{x^1}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \cdots$$

iii. $k(0) = 1$

d. Solution for All Functions Preserving the Key Properties

*...Application of key property (**ii**):*

$0 < k(x)$

*...Application of key property (**i**):*

$\frac{k'(x)}{k(x)} = 1$

...Antidifferentiation with respect to x:

$\ln k(x) = x + \mathrm{c}$

$\ln k(0) = \mathrm{c}$

*...Application of key property (**iii**):*

$\ln 1 = \mathrm{c}$

...Substitution for ln definition:

$\int_1^1 \frac{1}{s} \mathrm{d}s = \mathrm{c}$

$0 = \mathrm{c}$

...Substitution for constant c back into previous equation:

... $\ln k(x) = x$

- $k(x) = \exp x$
- $\exp x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

8. Conclusion

$$\begin{aligned}
2^\pi &= \exp(\pi \ln 2) \\
&= 1 \\
&\quad + \left((2\pi/1)(1/3)^1 + (2\pi/3)(1/3)^3 + (2\pi/5)(1/3)^5 + (2\pi/7)(1/3)^7 + \cdots\right)^1/1 \\
&\quad + \left((2\pi/1)(1/3)^1 + (2\pi/3)(1/3)^3 + (2\pi/5)(1/3)^5 + (2\pi/7)(1/3)^7 + \cdots\right)^2/(1\cdot 2) \\
&\quad + \left((2\pi/1)(1/3)^1 + (2\pi/3)(1/3)^3 + (2\pi/5)(1/3)^5 + (2\pi/7)(1/3)^7 + \cdots\right)^3/(1\cdot 2\cdot 3) \\
&\quad + \left((2\pi/1)(1/3)^1 + (2\pi/3)(1/3)^3 + (2\pi/5)(1/3)^5 + (2\pi/7)(1/3)^7 + \cdots\right)^4/(1\cdot 2\cdot 3\cdot 4) \\
&\quad + \cdots \\
&\approx 8.824977827
\end{aligned}$$

The expression 2^π is meaningful because the generalization preserves the three defining properties that populate the simple exponent.

9. Appendix: Proof by Excel Spreadsheet

a. Formula Input: Copy With Formatting (ctrl + h replacing all "≈" with "=" produces formula output)

output	a	input		$\ln(a)$	a^input	exp(input $\ln(a)$)
calculator:	≈2	≈PI()		≈LN(B2)	≈B2^C2	≈EXP(C2*LN(B2))
	a	input	i	$\ln(a)$	i!	exp(input $\ln(a)$)
manually:	≈2	≈PI()	0	≈-2/(D4*2+1)*((1-$B$4)/(1+$B$4))^(D4*2+1)	≈1	≈1
			1	≈E4-2/(D5*2+1)*((1-$B$4)/(1+$B$4))^(D5*2+1)	≈D5*F4	≈G4+($C$4*$E$23)^D5/F5
			2	≈E5-2/(D6*2+1)*((1-$B$4)/(1+$B$4))^(D6*2+1)	≈D6*F5	≈G5+($C$4*$E$23)^D6/F6
			3	≈E6-2/(D7*2+1)*((1-$B$4)/(1+$B$4))^(D7*2+1)	≈D7*F6	≈G6+($C$4*$E$23)^D7/F7
			4	≈E7-2/(D8*2+1)*((1-$B$4)/(1+$B$4))^(D8*2+1)	≈D8*F7	≈G7+($C$4*$E$23)^D8/F8
			5	≈E8-2/(D9*2+1)*((1-$B$4)/(1+$B$4))^(D9*2+1)	≈D9*F8	≈G8+($C$4*$E$23)^D9/F9
			6	≈E9-2/(D10*2+1)*((1-$B$4)/(1+$B$4))^(D10*2+1)	≈D10*F9	≈G9+($C$4*$E$23)^D10/F10
			7	≈E10-2/(D11*2+1)*((1-$B$4)/(1+$B$4))^(D11*2+1)	≈D11*F10	≈G10+($C$4*$E$23)^D11/F11
			8	≈E11-2/(D12*2+1)*((1-$B$4)/(1+$B$4))^(D12*2+1)	≈D12*F11	≈G11+($C$4*$E$23)^D12/F12
			9	≈E12-2/(D13*2+1)*((1-$B$4)/(1+$B$4))^(D13*2+1)	≈D13*F12	≈G12+($C$4*$E$23)^D13/F13
			10	≈E13-2/(D14*2+1)*((1-$B$4)/(1+$B$4))^(D14*2+1)	≈D14*F13	≈G13+($C$4*$E$23)^D14/F14
			11	≈E14-2/(D15*2+1)*((1-$B$4)/(1+$B$4))^(D15*2+1)	≈D15*F14	≈G14+($C$4*$E$23)^D15/F15
			12	≈E15-2/(D16*2+1)*((1-$B$4)/(1+$B$4))^(D16*2+1)	≈D16*F15	≈G15+($C$4*$E$23)^D16/F16
			13	≈E16-2/(D17*2+1)*((1-$B$4)/(1+$B$4))^(D17*2+1)	≈D17*F16	≈G16+($C$4*$E$23)^D17/F17
			14	≈E17-2/(D18*2+1)*((1-$B$4)/(1+$B$4))^(D18*2+1)	≈D18*F17	≈G17+($C$4*$E$23)^D18/F18
			15	≈E18-2/(D19*2+1)*((1-$B$4)/(1+$B$4))^(D19*2+1)	≈D19*F18	≈G18+($C$4*$E$23)^D19/F19
			16	≈E19-2/(D20*2+1)*((1-$B$4)/(1+$B$4))^(D20*2+1)	≈D20*F19	≈G19+($C$4*$E$23)^D20/F20
			17	≈E20-2/(D21*2+1)*((1-$B$4)/(1+$B$4))^(D21*2+1)	≈D21*F20	≈G20+($C$4*$E$23)^D21/F21
			18	≈E21-2/(D22*2+1)*((1-$B$4)/(1+$B$4))^(D22*2+1)	≈D22*F21	≈G21+($C$4*$E$23)^D22/F22
			19	≈E22-2/(D23*2+1)*((1-$B$4)/(1+$B$4))^(D23*2+1)	≈D23*F22	≈G22+($C$4*$E$23)^D23/F23

b. Formula Output

output	a	input		$\ln(a)$	a^input	exp(input $\ln(a)$)
calculator:	2	3.141592654		0.693147181	8.824977827	8.824977827
	a	input	i	$\ln(a)$	i!	exp(input $\ln(a)$)
manually:	2	3.141592654	0	0.666666667	1	1
			1	0.691358025	1	3.17758609
			2	0.693004115	2	5.548526681
			3	0.693134757	6	7.269502431
			4	0.693146047	24	8.206395645
			5	0.693147074	120	8.614428771
			6	0.69314717	720	8.762516647
			7	0.69314718	5040	8.808584376
			8	0.69314718	40320	8.821123932
			9	0.693147181	362880	8.824157927
			10	0.693147181	3628800	8.824818606
			11	0.693147181	39916800	8.824949396
			12	0.693147181	479001600	8.824973129
			13	0.693147181	6227020800	8.824977105
			14	0.693147181	87178291200	8.824977723
			15	0.693147181	1.30767E+12	8.824977813
			16	0.693147181	2.09228E+13	8.824977825
			17	0.693147181	3.55687E+14	8.824977827
			18	0.693147181	6.40237E+15	8.824977827
			19	0.693147181	1.21645E+17	8.824977827